HIGHLY CONNECTED MANIFOLDS OF POSITIVE p-CURVATURE

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Abstract. We study and in some cases classify highly connected manifolds which admit a Riemannian metric with positive p-curvature. The p-curvature was defined and studied by the second author in [8, 9, 10]. It turns out that positivity of p-curvature could be preserved under surgeries of codimension at least $p+3$. This gives a key to reduce a geometrical classification problem to a topological one, in terms of relevant bordism groups and index theory. In particular, we classify 3-connected manifolds with positive 2-curvature in terms of the bordism groups Ω_*^{spin}, $\Omega_*^{\mathrm{string}}$, and by means of α-invariant and Witten genus ϕ_W. Here we use results from [5], which provide appropriate generators of the ring $\Omega_*^{\mathrm{string}} \otimes \mathbf{Q}$ in terms of "geometric $\mathbb{C}\mathbf{aP}^2$-bundles", where the Cayley projective plane $\mathbb{C}\mathbf{aP}^2$ is a fiber and the structure group is F_4 which is the isometry group of the standard metric on $\mathbb{C}\mathbf{aP}^2$.

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1. Introduction and statement of the main results

1.1. **Positive scalar curvature.** There is a fundamental result due to Gromov and Lawson [6], Schoen and Yau [17] known as "Surgery Theorem". It shows that positivity of the scalar curvature could be preserved under surgery of codimension at least three. In particular, the surgery technique provides a key to classify simply-connected manifolds admitting a metric with positive scalar curvature, [6], [18]. These are the results:

Theorem 1.1. [6, Theorem A] *Let M be a compact non-spin simply-connected manifold, $\dim M = n \geq 5$. Then M always admits a metric g with positive scalar curvature.*

Let $\alpha : \Omega_n^{\mathrm{spin}} \to KO_n$ be the Atiyah-Bott-Shapiro homomorphism which evaluates the index of the Dirac operator on a spin manifold M representing a cobordism class $[M] \in \Omega_n^{\mathrm{spin}}$.

Theorem 1.2. [6, Theorem B], [18] *Let M be a compact spin simply-connected manifold, $\dim M = n \geq 5$. Then M admits a metric g with positive scalar curvature if and only if $\alpha([M]) = 0$ in the group KO_n.*

It turns out that there are many other Riemannian invariants that are also stable under some type of surgeries, see, for example, [1, 4, 13, 14, 20]. Among such invariants are p-curvature s_p and the second Gauss-Bonnet curvature which were studied by the second author, see [8, 11].

1.2. **Positive p-curvature.** Let (M, g) be a Riemannian manifold, and TM be the tangent bundle. We denote by $G_p(TM)$ the bundle of Grassmanians of p-dimensional subspaces of the tangent bundle TM. Then the p-curvature s_p is defined as a function $s_p : G_p(TM) \to \mathbf{R}$ as follows. For a p-dimensional space $V \in G_p(TM_x)$, the value of $s_p(V)$ is a "partial trace" of the curvature tensor, along all directions, perpendicular to the subspace $V \subset TM_x$, see [8] and

section 5 for details. The curvature s_0 is nothing but the scalar curvature Scal, furthermore, the function s_{p-1} could be thought as an appropriate trace of s_p. In particular, positivity of s_p implies positivity of the curvatures s_j for all $j < p$, including the scalar curvature. It turns out that positivity of the p-curvature s_p is also stable under surgeries of codimension at least $3+p$, see [8, Main Theorem].

The surgery result [8, Main Theorem] gives an appropriate setup to classify manifolds admitting a metric with positive p-curvature for $p \geq 1$ similar to the case of positive scalar curvature. The first interesting case is when $p = 1$. Then the curvature s_1 coincides (up to a factor 2) with the quadratic form associated to the 2-tensor S defined by

$$S_{ij} = \tfrac{1}{2}\mathrm{Scal} \cdot g_{ij} - \mathrm{Ric}_{ij}.$$

The tensor $-S$ is also known as the *Einstein tensor*, and the 1-curvature s_1 is called the *Einstein curvature*, see [8] and [9]. We notice that

$$\operatorname{tr} S = \tfrac{(n-2)}{2}\mathrm{Scal}\ .$$

Thus positivity of the tensor S is the same as positivity of the curvature s_1, and positivity of s_1 implies positivity of the scalar curvature for $n \geq 3$.

An interesting case here is when the manifolds in question are 2-connected. Then such manifolds are necessarily spin-manifolds, and the relevant cobordism group is Ω_n^{spin}. Here is the classification result analogous to Theorem 1.2:

Theorem 1.3. [8, Theorem I] *Let M be a compact 2-connected manifold, $\dim M = n \geq 7$. Then M admits a metric g with positive 1-curvature if and only if $\alpha([M]) = 0$ in the group KO_n.*

The main technique to prove Theorem 1.3, is a Surgery Theorem [8, Main Theorem] and the results by S. Stolz on *geometric* $\mathbf{HP}^2$*-bundles.*

1.3. **Geometric $\mathbf{HP}^2$-bundles.** We recall that in order to prove that vanishing of the index $\alpha([M]) \in KO_n$ is sufficient for existence of a metric with positive scalar curvature on M, S. Stolz proves that all cobordism classes in $\ker\alpha \subset \Omega_n^{\mathrm{spin}}$ could be realized as total spaces of geometric $\mathbf{HP}^2$-bundles.

In more detail, let $PSp(3)$ be the projectivization of the symplectic orthogonal group $Sp(3)$. It is well-known that the group $PSp(3)$ is the isometry group of the standard metric on $\mathbf{HP}^2$. Let $BPSp(3)$ be the classifying space of the group $PSp(3)$, and $EPSp(3) \to BPSp(3)$ be the universal principal bundle. We obtain the *universal geometric* $\mathbf{HP}^2$*-bundle* $E(\mathbf{HP}^2) \to BPSp(3)$ with a fiber $\mathbf{HP}^2$ and a structure group $PSp(3)$, where the total space $E(\mathbf{HP}^2)$ is defined in a usual way:

$$E(\mathbf{HP}^2) = EPSp(3) \times_{PSp(3)} \mathbf{HP}^2.$$

Then for any map $f : B \to BPSp(3)$, there is a natural pull-back $\mathbf{HP}^2$-bundle $E \to B$ given by the diagram:

$$\begin{array}{ccc} E & \xrightarrow{\hat{f}} & E(\mathbf{HP}^2) \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BPSp(3) \end{array}$$

This construction defines a transfer map

$$T : \Omega^{\mathrm{spin}}_{n-8}(BPSp(3)) \longrightarrow \Omega^{\mathrm{spin}}_{n}$$

which takes a cobordism class of a map $f : B \to BPSp(3)$ to the cobordism class of the manifold E as above. The following result provides a key to prove the necessity in Theorem 1.2.

Theorem 1.4. (Stolz, [18]) *There is an isomorphism* $\mathrm{Im}\, T \cong \ker \alpha$.

By construction, a total space E of a geometric $\mathbf{HP}^2$-bundle carries a metric with positive scalar curvature, which is given by a choice of any metric on the base and giving a standard homogeneous metric to each fiber $\mathbf{HP}^2$ scaled appropriately to get a positive scalar curvature on E.

One can observe that if E is a total space of a geometric $\mathbf{HP}^2$-bundle, then it carries a metric with positive curvature s_p for $p \leq 6$.

1.4. **Main results.** Assume that M is a 3-connected manifold, then M has a canonical spin-structure. There are two possibilities: either M is string-manifold or not. It is well-known that the obstruction to existence of a string-structure is given by $\frac{1}{2}p_1(M)$, where p_1 is the first Pontryagin class. The following result is somewhat analogous to Theorem 1.1:

Theorem A. *Let M be a compact 3-connected non-string manifold with* $\dim M = n \geq 7$. *Then M admits a Riemannian metric g with positive 2-curvature if and only if* $\alpha([M]) = 0$ *in the group KO_n, where $\alpha : \Omega^{\mathrm{spin}}_n \to KO_n$ is as above.*

Any 3-connected manifold is spin, and Ω^{spin}_* is a relevant bordism group here. We use surgery technique and Theorem 1.4 to show that if $\alpha([M]) = 0$ and M is not string, then it has a metric with positive 2-curvature which is "pulled back" from a nice metric on a total space of a geometric $\mathbf{HP}^2$-bundle as above.

Let now M be 3-connected and string then a relevant bordism group here is $\Omega^{\mathrm{string}}_*$. Precisely, we prove the following theorem which is analogous to Theorem B of [6].

Theorem B. *Let M_1 be a compact $(3+r)$-connected, $0 \leq r \leq 3$, string manifold of dimension $n \geq 9+2r$. Assume that $[M_1] = [M_0]$ in the cobordism group $\Omega^{\mathrm{string}}_n$, where M_0 admits a metric g_0 with $s_{r+2}(g_0) > 0$. Then M_1 also admits a metric g_1 with $s_{r+2}(g_1) > 0$.*

In particular, a compact 3-connected string manifold M of dimension $n \geq 9$ that is string cobordant to a manifold of positive 2-curvature admits a metric with positive 2-curvature.

For instance, if M is string cobordant to zero, then the conclusion of the theorem holds for M. It is known that $\Omega_n^{\text{string}} = 0$ for $n = 11$ or $n = 13$; therefore any compact 3-connected string manifold of dimension 11 or 13 always has a metric with positive 2-curvature.

Let I denote the subset of Ω_*^{string} which consists of bordism classes containing representatives with positive 2-curvature. Clearly I is an ideal of Ω_*^{string} since the cartesian product of a manifold of positive 2-curvature with an arbitrary manifold has positive 2-curvature. We therefore define the following *geometrical genus*:

$$\Pi : \Omega_*^{\text{string}} \to \Omega_*^{\text{string}}/I,$$

which is a ring homomorphism.

Let $\phi_W : \Omega_*^{\text{string}} \to \mathbf{Z}[[q]]$ be the Witten genus, see [5, 19], and section 6 below. By definition, if $\phi_W x \neq 0$, then $x \in \Omega_*^{\text{string}}$ has infinite order. We prove the following result which is analogous to Corollary B of [6].

Corollary B. *Let N be a $(3+r)$-connected, for $0 \leq r \leq 3$, string manifold of dimension at least $9+2r$ with vanishing Witten genus then some multiple $N\sharp\cdots\sharp N$ carries a metric of positive $(r+2)$-curvature.*

In particular, if N is a 3-connected string manifold of dimension at least 9 with vanishing Witten genus then some multiple $N\sharp\cdots\sharp N$ carries a metric of positive 2-curvature.

This result suggests that the geometric genus Π is related to Witten genus. It is an open question to prove that N itself carries a metric of positive 2-curvature.

Clearly, Theorems A, B and Corollary B give only partial classification of manifolds with metrics of positive 2-curvature. However, we use a construction which eventually may be useful to obtain an affirmative classification. Before stating our conjecture, we briefly describe the construction.

Let F_4 be the 52-dimensional compact simple sporadic Lie group. It is well-known that it contains a closed subgroup isomorphic to $Spin(9)$ which is unique up to inner automorphism. We denote by $\mathbb{C}\mathbf{aP}^2$ the Cayley projective plane which coincides with the homogeneous space $F_4/Spin(9)$. Then the canonical homogeneous metric on $\mathbb{C}\mathbf{aP}^2$ has F_4 as a full isometry group, see [21, p. 264]. Let BF_4 be a classifying space, and $EF_4 \to BF_4$ be a universal principle F_4-bundle. A universal *geometrical* $\mathbb{C}\mathbf{aP}^2$*-bundle* could be identified with the fiber bundle $BSpin(9) \to BF_4$ which has a fiber $\mathbb{C}\mathbf{aP}^2$ and a structure group F_4. Then for a manifold L and a map $f : L \to BF_4$, we obtain the following map of fiber bundles

$$\begin{tikzcd} W \arrow[r, "f^*"] \arrow[d, "\pi"] & BSpin(9) \arrow[d] \\ L \arrow[r, "f"] & BF_4 \end{tikzcd} \tag{1}$$

The fiber bundle $\pi : W \to L$ as above is called a *geometrical* $\mathbb{C}\mathbf{aP}^2$*-bundle.*

It is well-known that $\Omega_*^{\rm string}\otimes \mathbf{Q}$ is a polynomial ring. In more detail, A. Dessai shows that there exist generators x_{4k}, such that

$$\Omega_*^{\rm string}\otimes \mathbf{Q} \cong \mathbf{Q}[x_8, x_{12}, x_{16}, \ldots] \ ,$$

and each element x_{4k} with $k\geq 4$ is represented by a manifold W^{4k} which is a total space of a geometrical $\mathbb{C}\mathbf{aP}^2$-bundle $\pi_k : W^{4k} \to L^{4k-16}$, see [5] and section 6 below. We consider a transfer map

$$T^{\rm string} : \Omega_{\ell}^{\rm string}(BF_4) \longrightarrow \Omega_{\ell+16}^{\rm string}$$

given as follows. Let $f: L \to BF_4$ be a map representing an element $x\in \Omega_{\ell}^{\rm string}(BF_4)$. Then the manifold W from (2) represents the element $T^{\rm string}(x)\in \Omega_{\ell+16}^{\rm string}$. Also, we recall that there is an integral version of the Witten genus

$$\phi_W^{\mathbf{Z}} : \Omega_*^{\rm string} \to KO_*[[q]]$$

which factors through the coefficients tmf_* of the *topological modular forms theory* tmf (formally known as eo_2):

$$\begin{tikzcd} \Omega_*^{\rm string} \arrow[rr, "\phi_W^{\mathbf{Z}}"] \arrow[dr, "\phi_{AHR}"'] & & KO_*[[q]] \\ & \mathrm{tmf}_* \arrow[ur, "\omega"'] & \end{tikzcd} \tag{2}$$

Here $\phi_{AHR} : \Omega_*^{\rm string} \to \mathrm{tmf}_*$ is the string-orientation constructed by Ando, Hopkins and Strickland, see [2, 3].

Remark. It is known that the groups $\Omega_*^{\rm string}$ have no p-torsion away from $p=2,3$. It is tempting to conjecture that Im $T^{\rm string}$ and Ker ϕ_{AHR} coincide in $\Omega_*^{\rm string}$ localized at primes 2 and 3. It turns out, this is too optimistic: the authors were informed by M. Joachim that the image Im $T^{\rm string}$ is strictly less than Ker ϕ_{AHR} in dimension 32. Nevertheless, we think that one may use other homogeneous spaces, besides $\mathbb{C}\mathbf{aP}^2$, to represent elements of the kernel Ker ϕ_{AHR} by manifolds with positive 2-curvature.

Conjecture C. *Let M be a 3-connected string manifold with $\dim M = n \geq 9$. Then M admits a Riemannian metric of positive 2-curvature if and only if $\phi_{AHR}([M])=0$ in tmf_n.*

We note that Conjecture C is weaker than Stolz' conjecture [19, Conjecture 1.1] on the existence of a metric with positive Ricci curvature. However it seems that it is still very difficult to verify.

1.5. **Generalizations.** The previous results are generalized in this paper in different directions.

On one hand, we show that all the previous theorems and conjectures are still valid if one replaces everywhere positive 2-curvature s_2 by positive second Gauss-Bonnet curvature h_4 or by both $s_2>0$ and $h_4>0$. Recall that the h_4 curvature is a scalar function defined on the manifold that generalizes the usual scalar curvature. It is shown in [11] that it is preserved under surgeries of codimension at least 5.

On the other hand, we prove that similar results hold for 3-curvature s_3 in the frame of 4-connected Fivebrane and non Fivebrane manifolds. Recall that a Fivebrane manifold is a string manifold for which the fractional pontryagin class $\frac{1}{6}p_2$ vanishes.

The above Corollary B asserts in particular that if a compact 6-connected manifold N is with dimension ≥ 15 and with vanishing Witten genus then some multiple of it $N\sharp\cdots\sharp N$ carries a metric of positive 5-curvature. We prove the following analogous of Theorem A in this context:

Theorem A′. *Let N be a 7-connected and non-Fivebrane compact manifold of dimension ≥ 15. If N is string-cobordant to a manifold M which carries a metric with positive 6-curvature, then N also carries a metric with positive 6-curvature.*

In particular, if a compact non-Fivebrane 7-connected manifold N of dimension ≥ 15 has a vanishing Witten genus then some multiple of it $N\sharp\cdots\sharp N$ carries a metric of positive 6-curvature.

It remains an open question to prove that N itself carries a metric of positive 6-curvature.

From another prospective, we prove the following generalization of Theorem B:

Theorem B′. *Let M_1 be a compact $(4+r)$-connected, $0 \leq r \leq 3$, Fivebrane manifold of dimension $n \geq 11+2r$. Assume that $[M_1]=[M_0]$ in the cobordism group $\Omega_n^{\text{Fivebrane}}$, where M_0 admits a metric g_0 with $s_{3+r}(g_0)>0$. Then M_1 also admits a metric g_1 with $s_{3+r}(g_1)>0$.*

In particular, a compact 4-connected Fivebrane manifold M with $\dim M = n \geq 11$ that is Fivebrane cobordant to a manifold of positive 3-curvature also carries a metric with positive 3-curvature.

The paper also contains further generalizations of the previous results, whenever it is appropriate, to all higher p-curvatures in the case of highly connected $BO\langle\ell\rangle$-manifolds.

1.6. **Plan of the paper.** Section 2 contains basic definitions of string and Fivebrane manifolds, string and Fivebrane cobordism rings and more general $BO\langle\ell\rangle$-manifolds and the corresponding cobordism rings.

In sections 3 and 4, we prepare for the proof of the main results. In section 3, we study some interactions between the codimension size of a surgery made within a given $BO\langle\ell\rangle$-cobordism class and the order of connectivity of representatives of that class. In section 4, we recall the definitions of p-curvatures s_p and the second Gauss-Bonnet curvature h_4. We emphasize on the most important property of these curvatures precisely the stability of their positivity under surgeries of sufficiently high codimensions.

In section 5, we prove Theorems A, B and B′. In section 6 we recall useful material about the Witten genus and the recent results of Dessai about the rational cobordism groups and the Kernel of the Witten genus. The results of section 6 are used in section 7 to prove Theorem A′ and Corollary B.

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2. String and $BO\langle\ell\rangle$-cobordism: Basic definitions

Let $\mathbf{R}^{n+k}$ be the Euclidian space. We denote by $G_k(\mathbf{R}^{n+k})$ the Grassmanian manifold of n-dimensional subspaces of $\mathbf{R}^{n+k}$, and by

$$U_{k,n} \longrightarrow G_k(\mathbf{R}^{n+k}) \quad \text{and} \quad U_{k,n}^{\perp} \longrightarrow G_k(\mathbf{R}^{n+k})$$

the tautological bundle and its orthogonal complement respectively. Then one obtains the spaces

$$BO(n) := \lim_k G_k(\mathbf{R}^{n+k}), \quad \text{and} \quad BO := \lim_n BO(n)$$

which are the classifying spaces of the orthogonal group $O(n)$ and its stable version $O := \lim\limits_n O(n)$. The homotopy groups of BO are well-known:

$$\pi_q BO = \begin{cases} \mathbf{Z}/2 & \text{if } q = 1,2 \pmod 8 \\ \mathbf{Z} & \text{if } q = 0,4 \pmod 8 \\ 0 & \text{else} \end{cases}$$

Consider the Postnikov tower of the space BO:

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ BO\langle 8\rangle & \xrightarrow{p_2/6} & K(\mathbf{Z},8) \\ \downarrow & & \\ BSpin & \xrightarrow{p_1/2} & K(\mathbf{Z},4) \\ \downarrow & & \\ BSO & \xrightarrow{w_2} & K(\mathbf{Z}/2,2) \\ \downarrow & & \\ BO & \xrightarrow{w_1} & K(\mathbf{Z}/2,1) \end{array} \tag{3}$$

In each step the lowest homotopy group is killed by the map into the corresponding Eilenberg-McLane space, and w_1, w_2 are the Stiefel-Whitney classes and p_1, p_2 are the Pontryagin classes respectively.

Now let M be a manifold, $\dim M = n$. We denote by h_0 the Euclidian metric on $\mathbf{R}^{n+k}$. Then an embedding $j : M \hookrightarrow \mathbf{R}^{n+k}$ provides M with the Riemannian metric $g = j^* h_0$ induced from the Euclidian space $\mathbf{R}^{n+k}$. Furthemore, the metric g gives the tangent and normal bundles TM and NM the Euclidian structure, in particular, we have the Gauss map

$$\bar{f} : M \longrightarrow G_k(\mathbf{R}^{n+k})$$

such that $\bar{f}^*U_{k,n} = NM$ and $\bar{f}^*U_{k,n}^{\perp} = TM$. A homotopy class of $\bar{f}$ depends on the embedding $j : M \hookrightarrow \mathbf{R}^{n+k}$, however, it determines uniquely a homotopy class of the composition

$$f : M \xrightarrow{\bar{f}} G_k(\mathbf{R}^{n+k}) \longrightarrow BO.$$

We say that a manifold M has a *string*-structure if the Gauss map $f : M \longrightarrow BO$ lifts to the map $\hat{f} : M \longrightarrow BO\langle 8\rangle$, i.e. the following diagram commutes:

$$\begin{tikzcd}
 & BO\langle 8\rangle \arrow[d] & \\
 & BSpin \arrow[d] \arrow[r, "p_1/2"] & K(\mathbf{Z},4) \\
 & BSO \arrow[d] \arrow[r, "w_2"] & K(\mathbf{Z}/2,2) \\
M \arrow[r, "f"] \arrow[ur] \arrow[uur] \arrow[uuur, "\hat{f}"] & BO \arrow[r, "w_1"] & K(\mathbf{Z}/2,1)
\end{tikzcd} \tag{4}$$

A choice of the lift $\hat{f}$ is sometimes called a *string-structure* on M. We emphasize that usually we use a string-structure on the normal bundle NM; this implies that the tangent bundle TM also has a string structure. We denote by $\Omega_n^{\mathrm{string}}$ the corresponding cobordism group.

This construction has more general setting. Let $BO\langle \ell\rangle$ be the $(\ell-1)$-connected cover of BO. We say that a manifold M has $BO\langle \ell\rangle$-structure if it is given a lift $f_{\langle \ell\rangle}$ of the standard Gauss map as above. Then there is a corresponding cobordism group $\Omega_n^{\langle \ell\rangle}$. Clearly we have that $\Omega_n^{\langle 4\rangle} = \Omega_n^{\mathrm{spin}}$, and $\Omega_n^{\langle 8\rangle} = \Omega_n^{\mathrm{string}}$. There is one more special case when manifolds have $BO\langle 9\rangle$-structure: these are string manifolds with the vanishing class $\frac{1}{6}p_2$. In some papers, for instance see [16], manifolds with $BO\langle 9\rangle$-structure are called as *Fivebrane manifolds*, and the cobordism group $\Omega_n^{\langle 9\rangle}$ is called *Fivebrane cobordism* and denoted as $\Omega_n^{\mathrm{Fivebrane}}$.

3. Surgeries and $BO\langle \ell\rangle$-manifolds

Let M be a closed n-manifold, $S^s \subset M$ be an embedded sphere with trivial normal bundle and let $t = n - s - 1$. This gives an embedding $S^s \times D^{t+1} \subset M$ which extends the embedding $S^s \subset M$. Then a *surgery along* the embedded $S^s \subset M$ gives the manifold

$$M' = \left(M \setminus (S^s \times D^{t+1})\right) \cup_{S^s \times S^t} (D^{s+1} \times S^t).$$

Let $x \in \pi_s(M)$ be an element represented by a map $\xi : S^s \to M$. If $2s < n$, then according to Whitney Embedding Theorem the map ξ could be deformed to an embedding $S^s \subset M$. Then we say that *the element $x \in \pi_s(M)$ could be killed by a surgery* if such an embedding has trivial normal bundle.

Let $f: M \to BO$ be the map classifying the stable normal bundle of M; it gives the induced homomorphism $f_*: \pi_s(M) \to \pi_s(BO)$. The following result is well-known (see, [15, Corollary 5.64], for example):

Lemma 3.1. *Let M be a smooth manifold, $f: M \to BO$ be the map classifying the stable normal bundle of M. Assume $2s < n = \dim M$. Then an element $x \in \pi_s(M)$ could be killed by a surgery if and only if $f_*(x) = 0$ in $\pi_s(BO)$.*

The previous lemma 3.1 implies in particular that for a $BO\langle\ell\rangle$-manifold M of dimension $n > 2s$, every map $S^s \to M$ has trivial normal bundle for $s \leq l-1$. We have therefore proved the following lemma:

Lemma 3.2. *Assume $n > 2(\ell - k) \geq 2$ and $k \geq 1$.*

(1) *Let $x \in \Omega_n^{\langle\ell\rangle}$, then x could be represented by some $(\ell-k)$-connected manifold.*

(2) *Let M_0 and M_1 be $(\ell-k)$-connected $BO\langle\ell\rangle$-manifolds. Then if M_0 and M_1 represent the same element $x \in \Omega_n^{\langle\ell\rangle}$, there exists a $BO\langle\ell\rangle$-cobordism (W, M_0, M_1), where the pair (W, M_1) is $(\ell-k)$-connected.*

In particular, if $n > 8$, any element $x \in \Omega_n^{\mathrm{string}}$ can be represented by a 4-connected manifold and if M_0, M_1 are two 4-connected manifolds representing x, then there exists a string-cobordism (W, M_0, M_1) where (W, M_1) is 4-connected.

Next, we need more details on bordisms between $BO\langle\ell\rangle$-manifolds. We start with the following fact which follows from the basic Morse theory:

Lemma 3.3. *Let (W, M_0, M_1) be a simply connected bordism, $\dim W = n+1$, and let $n \geq p+3$, where p is a positive integer. Assume $H_j(W, M_1; \mathbf{Z}) = 0$ for all $j \leq p$, then M_1 can be obtained from M_0 by surgeries of codimension at least $p+1$.*

The following result is a consequence of Lemma 3.3:

Proposition 3.4. *Let M_1 be a compact r-connected $BO\langle\ell\rangle$-manifold of dimension n, where $n \geq 2r+3$ and $\ell \geq r+2$. Let M_0 be a compact manifold, such that $[M_0] = [M_1]$ in $\Omega_n^{\langle\ell\rangle}$ [1]. Then M_1 can be obtained from M_0 by surgeries of codimension at least $r+2$.*

Proof. Let (W, M_0, M_1) be a $BO\langle\ell\rangle$-cobordism, and M_1 is r-connected. Using surgeries we can assume that W is $(r+1)$-connected since $r+1 \leq \ell - 1$ and the dimension of W is sufficiently high. Consequently, we have $H_i(W) = 0$ for all $i \leq r+1$. On the other hand M_1 is r-connected, thus $H_i(M_1) = 0$ for all $i \leq r$ and therefore $H_i(W, M_1) = 0$ for all $i \leq r+1$. Finally, Lemma 3.3 implies that M_1 can be obtained from M_0 by surgeries of codimension $r+2$. □

[1] In particular, M_0 is a $BO\langle\ell\rangle$- manifold.

Remarks 3.1. (1) For $r = 1$ and $\ell = 4$, Proposition 3.4 asserts the following: if M_1 is compact simply connected spin manifold of dimension $n \geq 5$ that is spin-cobordant to a manifold M_0, then M_1 can be obtained from M_0 by surgeries of codimension at least three. This was first noticed by Gromov and Lawson, see [6].

(2) For $r = 2$ and $\ell = 4$, Proposition 3.4 is the same as surgery Lemma 4.2 in [8]: if M_1 is compact 2-connected manifold of dimension $n \geq 7$ that is spin-cobordant to a manifold M_0, then M_1 can be obtained from M by surgeries of codimension ≥ 4.

We specify Theorem 3.4 for string-manifolds.

Corollary 3.5. *Let M_1 be a compact r-connected string manifold of dimension $n \geq 2r+3$, where $r \leq 6$. Then if M_0 is a compact manifold such that $[M_0] = [M_1]$ in $\Omega_n^{\mathrm{string}}$, then M_1 can be obtained from M_0 by surgeries of codimensions at least $r+2$.*

In particular, if M_1 is a compact 3-connected string manifold of dimension $n \geq 9$ string-cobordant to a manifold M_0, then M_1 can be obtained from M_0 by surgeries of codimensions at least 5.

If we continue our climbing to the Postnikov tower we reach to $BO\langle 9\rangle$-manifolds or *Fivebrane manifolds*. It is well known that the corresponding Postnikov invariant is given by $\frac{1}{6}p_2$, where p_2 is the second Pontryagin class. We specify Proposition 3.4 for Fivebrane-manifolds:

Corollary 3.6. *Let M_1 be an r-connected Fivebrane manifold of dimension $n \geq 2r+3$, where $r \leq 7$. Then if M_0 is a Fivebrane-manifold with $[M_0] = [M_1]$ in $\Omega_n^{\mathrm{Fivebrane}}$, then M_1 can be obtained from M_0 by surgeries of codimensions at least $r+2$.*

In particular, if M_1 is a compact 4-connected Fivebrane manifold of dimension $n \geq 11$ that is Fivebrane-cobordant to a manifold M_0, then M_1 can be obtained from M_0 by surgeries of codimensions at least 6.

3.1. **Non-string 3-connected manifolds.** In contrast with the result in Corollary 3.5, we prove the following result for 3-connected but non-string manifolds.

Proposition 3.7. *Let M_1 be a 3-connected and non-string compact manifold of dimension ≥ 7. If M_1 is spin cobordant to a manifold M_0, then M_1 can be obtained from M_0 by surgeries of codimension ≥ 5.*

Proof. Let (W, M_0, M_1) be a spin cobordism, where M_1 is 3-connected, non-string with dimension at least 7, and W is spin. Using surgeries we can assume that W is 3-connected as $\dim W = n+1 \geq 8$. Consequently by Hurewicz theorem we have $H_i(W) = 0$ for all $i = 1, 2, 3$ and $H_4(W) \cong \pi_4(W)$. Similarly, $H_i(M_1) = 0$ for $i = 1, 2, 3$ and $H_4(M_1) \cong \pi_4(M_1)$ since M_1 is 3-connected.

Since for any 3-connected space X, $H^4(X; \mathbf{Z}) \cong \mathrm{Hom}(H_4(X; \mathbf{Z}), \mathbf{Z})$, the first Pontryagin class $p_1(W)$ is given by a homomorphism

$$p_1(W) : H_4(W; \mathbf{Z}) \longrightarrow \mathbf{Z}.$$

Similarly, the class $p_1(M_1)$ is given by a homomorphism $p_1(M_1): H_4(M_1;\mathbf{Z}) \longrightarrow \mathbf{Z}$. Then $TW|_N \cong TM_1 \oplus \epsilon^1$, where ϵ^1 is a trivial linear bundle, which implies that $p_1(W) = i^*(p_1(M_1))$, where $i: M_1 \hookrightarrow W$ is the boundary inclusion. Also recall that the first Pontryagin class is divisible by 2 for spin manifolds. Thus we obtain a commutative diagram:

$$
\begin{tikzcd}
\pi_4(M_1) \cong H_4(M_1) \arrow[rr, "\frac{1}{2}p_1(M_1)"] \arrow[dr, "i_*"] & & \mathbf{Z} \\
& \pi_4(W) \cong H_4(W) \arrow[ur, "\frac{1}{2}p_1(W)"'] &
\end{tikzcd}
\tag{5}
$$

Remark. Let $H_4(W;\mathbf{Z}) = F_4(W) \oplus T_4(W)$, where $F_4(W)$ and $T_4(W)$ are free and torsion parts respectively. Clearly the homomorphism $p_1(W): H_4(W;\mathbf{Z}) \longrightarrow \mathbf{Z}$, restricted to the torsion part $T_4(W)$, is trivial. Thus $p_1(W)$ is not a torsion class, and $p_1(W)/2 = 0$ implies $p_1(W) = 0$.

Both manifolds W and M_1 are 3-connected, spin (where, of course, $M_1 \subset \partial W$), however, they are not string-manifolds, i.e. the first Pontryagin class is not zero. We would like to show that the kernel of the homomorphism $p_1/2$ could be killed by surgeries.

Lemma 3.8. *Let $\eta^k \to S^4$ be a vector bundle of dimension $k \geq 5$. Then there exists a 4-dimensional bundle $\xi^4 \to S^4$ such that $\eta^k \cong \xi^4 \oplus \epsilon^{k-4}$, where $\epsilon^{k-4} \to S^4$ is a trivial vector bundle.*

Proof. Let $f: S^4 \to BO(k)$ be a map classifying the bundle η. Then we can assume that S^4 is mapped to the 4-th skeleton $BO(k)^{(4)}$ of $BO(k)$. It is well-known that $BO(k)^{(4)} \subset BO(4)$ if $k \geq 5$. Thus up to homotopy, the map f factors through $BO(4)$, i.e. we obtain a commutative (up to homotopy) diagram:

$$
\begin{tikzcd}
S^4 \arrow[rr, "f"] \arrow[dr, "f_1"'] & & BO(k) \\
& BO(4) \arrow[ur, "\iota"'] &
\end{tikzcd}
$$

where $\iota: BO(4) \hookrightarrow BO(k)$ is the standard embedding. Hence $\eta^k \cong \xi^4 \oplus \epsilon^{k-4}$. □

We continue with the proof of Proposition 3.7. Let $S^4 \hookrightarrow M_1$ be an embedded sphere representing an element $x \in \pi_4(M_1) \cong H_4(M_1;\mathbf{Z})$ such that $p_1(x) = 0$. We denote by ν_{S^4} the normal of the embedding $S^4 \hookrightarrow M_1$. By assumption, $TM_1|_{S^4}$ is stably trivial, i.e. $TM_1|_{S^4} \oplus \epsilon^{k-n} \cong \epsilon^k$ for some $k > n$. However, $TM_1|_{S^4} \cong TS^4 \oplus \nu_{S^4}$, and we have that

$$TS^4 \oplus \nu_{S^4} \oplus \epsilon^{k-n} \cong \epsilon^k$$

Since $TS^4 \oplus \epsilon^1$ is a trivial bundle, thus

$$\epsilon^5 \oplus \nu_{S^4} \oplus \epsilon^{k-n-1} \cong \epsilon^k.$$

In particular, we obtain that $p_1\nu_{S^4} = 0$, and Lemma 3.8 gives that $\nu_{S^4} = \xi^4 \oplus \epsilon^{n-4}$ where ξ^4 is a 4-dimensional bundle with $p_1\xi^4 = 0$. Thus ξ^4 is trivial bundle, i.e. the normal bundle ν_{S^4}

is trivial. The same holds for an embedded spheres representing kernel of the homomorphism $p_1(W) : H_4(W;\mathbf{Z}) \longrightarrow \mathbf{Z}$.

Thus we can use surgeries on W to kill the kernel of $\frac{1}{2}p_1(W)$ and therefore the homomorphism $i_* : H_4(M_1) \to H_4(W)$ becomes an isomorphism onto its image. Therefore in the long exact sequence

$$H_4(M_1) \longrightarrow H_4(W) \longrightarrow H_4(W, M_1) \longrightarrow H_3(M_1)$$

the first map is onto and the last group is trivial then we conclude that $H_4(W, M_1) = 0$. Therefore we obtain that $H_i(W, M_1) = 0$ for all $i \leq 4$. Finally, Lemma 3.3 implies that M_1 can be obtained from M_0 by surgeries of codimension at least 5. □

3.2. **Non-Fivebrane 7-connected manifolds.** It is well-known that the second Pontryagin class of a string manifold is divisible by 6 and $\frac{1}{6}p_2$ serves as the obstruction to lifting a string structure to a Fivebrane structure. One can without difficulties adapt the proof of Proposition 3.7 to show the following:

Proposition 3.9. *Let N be a 7-connected and non-Fivebrane compact manifold of dimension ≥ 15. If N is string cobordant to a manifold M, then N can be obtained from M by surgeries of codimension ≥ 9.*

4. Positive curvature and Surgeries

The results of the previous section suggest that geometrical properties that are stable under surgeries should have a topological interpretation. This is the case for the positivity of the p-curvatures and the second Gauss-Bonnet curvature as we will see in the rest of this paper.

4.1. **Positive p-curvature.** We denote by $G_p(\mathbf{R}^n)$ the Grassmanian manifold of p-dimensional subspaces in $\mathbf{R}^n$, as above. Let (M, g) be a Riemannian manifold. Then the metric g provides the tangent bundle TM the structure group $O(n)$. This gives an associated smooth bundle

$$G_p(TM) := TM \times_{O(n)} G_p(\mathbf{R}^n) \longrightarrow M, \tag{6}$$

with the fiber $G_p(TM_x) \cong G_p(\mathbf{R}^n)$ over $x \in M$ and the structure group $O(n)$.

Then the p-curvature s_p, for $0 \leq p \leq n-2$, is a function

$$s_p : G_p(TM) \longrightarrow \mathbf{R}$$

defined as follows. Let V be a tangent p-plane at $x \in M$. We choose an orthonormal basis $\{e_i\}$ of the orthogonal complement $V^\perp$ of V in TM_x, and define

$$s_p(V) = \sum_{i,j=p+1}^{n} K_{i,j}, \tag{7}$$

where $K_{i,j} = K(e_i, e_j)$ is the usual sectional curvature. The 0-curvature s_0 coincides with the usual scalar curvature Scal, the 1-curvature is the Einstein curvature and the $(n-2)$-curvature is the usual sectional curvature.

We are interested under which conditions a manifold M admits a Riemannian metric g with positive p-curvature. We emphasize that if $s_p >$ then $s_j > 0$ for all $j < p$. It turns out that the positivity of the p-curvature could be preserved under surgeries:

Theorem 4.1 ([8]). *Let g_0 be a Riemannian metric on a compact manifold M_0 with $s_p(g_0) > 0$, and M_1 be a manifold constructed out of M_0 by a surgery of codimension $\geq p+3$. Then there exists a Riemannian metric g_1 on M_1 with $s_p(g_1) > 0$.*

There is a natural generalization of Theorem 4.1 for elementary cobordism:

Theorem 4.2. *Let g_0 be a Riemannian metric on a compact manifold M_0 with $s_p > 0$, and M_1 be a manifold constructed out of M_0 by a surgery of codimension $\ell + 1 \geq p+3$. Let*

$$W = M_0 \times I \cup (D^{k+1} \times D^{\ell+1}), \quad \partial W = M_0 \sqcup M_1,$$

be the corresponding elementary cobordism. Then there exists a Riemannian metric $\bar{g}$ on W with positive p-curvature and such that

$$\begin{cases} \bar{g} = g_0 + dt^2 & \text{near } M_0, \\ \bar{g} = g_1 + dt^2 & \text{near } M_1. \end{cases}$$

In particular, the p-curvature of the metric g_1 is positive.

4.2. **Positive second Gauss-Bonnet curvature.** For a given Riemannian manifold (M, g), we denote by R, Ric and Scal respectively the Riemann curvature tensor, the Ricci curvature tensor and the scalar curvature. The *second Gauss-Bonnet curvature*, denoted by h_4, is a quadratic scalar curvature and it is defined by

$$h_4 = ||R||^2 - ||\mathrm{Ric}||^2 + \tfrac{1}{4}\mathrm{Scal}^2,$$

see [11]. Note that in four dimensions, the curvature h_4 coincides with the Gauss-Bonnet integrand. This curvature is considered by physicists as a possible substitute to the usual scalar curvature to describe gravity in higher general theories of relativity, for instance in string theories. Here we are interested in the positivity properties of this invariant. First, let us recall the following stability under surgeries result:

Theorem 4.1 ([11]). *Let g_0 be a Riemannian metric on a compact manifold M_0 with $h_4(g_0) > 0$, and M_1 be a manifold constructed out of M_0 by a surgery of codimension at least 5. Then there exists a Riemannian metric g_1 on M_1 with $h_4(g_1) > 0$.*

There is a natural generalization of Theorem 4.1:

Theorem 4.2. *Let g_0 be a Riemannian metric on a compact manifold M_0 with $h_4(g_0) > 0$, and M_1 be a manifold constructed out of M_0 by a surgery of codimension $\ell + 1 \geq 5$. Let*

$$W = M_0 \times I \cup (D^{k+1} \times D^{\ell+1}), \quad \partial W = M_0 \sqcup M_1,$$

be the corresponding elementary cobordism. Then there exists a Riemannian metric $\bar g$ on W with $h_4(\bar g) > 0$ and such that

$$\begin{cases} \bar g = g_0 + dt^2 & \text{near } M_0, \\ \bar g = g_1 + dt^2 & \text{near } M_1. \end{cases}$$

In particular, $h_4(g_1) > 0$.

Remarks 4.3. (1) Theorems 4.1 and 4.2 are still valid if we require the metric g_0 to have positive h_4 and positive 2-curvature at the same time.

(2) Theorems 4.1 and 4.2 can be also be generalized to all higher Gauss-Bonnet curvatures h_{2k}. It is plausible that the condition $h_{2k} > 0$ could be preserved under surgeries of codimension at least $2k+1$.

(3) Because h_4 is quadratic in the Riemann curvature tensor, one can expect that the condition $h_4 > 0$ has two components, where each component is an inequality that is linear in the Riemann curvature tensor, it would be interesting to determine these components. In this direction, Theorem 4.3 below relates the positivity of h_4 to the positivity and negativity of the p-curvatures.

Theorem 4.3 ([11])**.** *Let (M, g) be a Riemannian manifold of dimension $n \geq 4$. Assume that $s_p(g) \geq 0$ or $s_p(g) \leq 0$ (respectively, $s_p(g) > 0$ or $s_p(g) < 0$), where $p \geq \frac{n}{2}$. Then $h_4(g) \geq 0$ (respectively, $h_4(g) > 0$). Furthermore, $h_4(g) \equiv 0$ if and only if the manifold (M, g) is flat.*

5. First Applications: Proof of Theorems A, B and B′

5.1. **Proof of theorem A.** Theorem A is a consequence of the following theorem

Theorem 5.1. *Let M_1 be a compact 3-connected manifold of dimension ≥ 7 which is not string. If M_1 is spin cobordant to a manifold M_0 which carries a metric g_0 with $s_2(g_0) > 0$ (respectively, with $h_4(g_0) > 0$), then M_1 also carries a metric g_1 with $s_2(g_1) > 0$ (respectively, with $h_4(g_1) > 0$).*

Proof. From one hand, Proposition 3.7 shows that the manifold M_1 can be obtained from M_0 by surgeries of codimension at least 5. On the other hand, since M_0 has positive 2-curvature (resp. positive h_4 curvature), Theorems 4.1 and 4.1 show therefore that M_1 also carries a metric with $s_2 > 0$ (respectively, with $h_4 > 0$). □

Now we prove Theorem A as follows. A compact 3-connected manifold M of positive 2-curvature and with dimension ≥ 7 is in particular a simply connected manifold of positive scalar curvature and therefore its α-genus vanishes by Theorem 1.2. Conversely, compact non-string 3-connected manifold M of dimension ≥ 7 and with vanishing α-genus is spin cobordant to the total space E of an $\mathbf{HP}^2$-bundle by Theorem 1.4. The total space E has positive 2-curvature and positive h_4 curvature then the above Theorem 5.1 shows that M carries a metric with positive 2-curvature and a metric with $h_4 > 0$.

5.2. **Proof of Theorems B and B′.** Now we return to $BO\langle\ell\rangle$-manifolds. The following theorem unifies and generalizes at the same time Theorem B of [6], Lemma 4.2 of [8] and Theorems B and B' of this paper that were stated in the introduction.

Theorem 5.2. *Let n, r, ℓ be positive integers such that $n \geq 2r+3$ and $\ell \geq r+2$. Let M_1 be a compact r-connected $BO\langle\ell\rangle$-manifold of dimension n. Assume $[M_1]=[M_0]$ in the cobordism group $\Omega_n^{\langle\ell\rangle}$, where M_0 admits a Riemannian metric g_0 with $s_p(g_0)>0$ for some p such that $0 \leq p \leq r-1$. Then M_1 also admits a Riemannian metric g_1 with $s_p(g_1)>0$.*

Proof. Let $0 \leq p \leq r-1$ be as in the theorem. Proposition 3.4 asserts that the manifold M_1 can be obtained from M_0 using surgeries of codimensions $\geq r+2 \geq p+3$. Since the manifold M_0 is supposed to have positive p-curvature then Theorem 4.1 shows that M_1 carries as well a metric with $s_p>0$. ☐

Note that we recover Theorem B of [6] about the scalar curvature (that is the 0-curvature) when $p=0, r=1$ and $\ell=4$. Lemma 4.2 of [8] about the 1-curvature is obtained for for $p=1, r=2$ and $\ell=4$.Theorems B and B' of this paper are respectively obtained for $\ell=8$ and $\ell=9$.

Remarks 5.1. (1) Similar results hold if we replace positive 2-curvature by positive h_4-scalar curvature, for instance: *A compact* 3*-connected string manifold of dimension $n \geq 9$ that is string cobordant to a manifold of positive h_4 has a metric with positive h_4 curvature.*

(2) Let $n=11$ or $n=13$. Since in these particular dimensions string n-manifolds are known to be cobordant to zero we conclude that a compact 3-connected string n-manifold always has a metric with positive 2-curvature and a metric with h_4 positive. Similar results hold for the p-curvatures for $p \leq 5$ as above.

5.3. **Genera for string manifolds and positive curvature.** Recall that the string cobordism ring $\Omega_*^{\text{string}} = \bigoplus_{n\geq 0} \Omega_n^{\text{string}}$ is is the ring whose elements of order n are string-cobordism classes of n-dimensional string manifolds, the addition operation is given by the disjoint union of manifolds and product operation is given by the Cartesian product of manifolds.

Let I_1 (resp. I_2, I_3) denote the subset of Ω_*^{string} which consists of bordism classes containing representatives with positive 2-curvature (resp. positive h_4, positive h_4 and positive s_2). Since the cartesian product of a manifold of positive 2-curvature (resp. positive h_4, positive h_4 and positive s_2) with an arbitrary manifold has positive 2-curvature (resp. positive h_4, positive h_4 and positive s_2), we conclude that I_1 (resp. I_2, I_3) is an ideal of Ω_*^{string}. We therefore get the following three genera (ring homomorphisms):

$$\Pi_i : \Omega_*^{\text{string}} \to \Omega_*^{\text{string}}/I_i, \tag{8}$$

for $i=1,2,3$. A natural question arises at this level: Are the previous three (geometrical) genera topological genera? Are they for instance related to Witten genus?

Recall that for a string manifold N, the Witten genus, denoted $\phi_W(N)$, is a modular form for $SL_2(\mathbf{Z})$ with integer coefficients. In particular, the Witten genus ϕ_W defines a ring homomorphism from the bordism ring Ω_*^{string} to the ring of integral modular forms for $SL_2(\mathbf{Z})$, see the next section for more details.

We shall prove in section 7 that Ker $\phi_W \otimes \mathbf{Q} \subset I_i$ for $i = 1, 2, 3$. It remains an open question to decide whether the previous inclusion is in fact an equality, that is a vanishing theorem of Lichnérowicz type: If N is a string manifold of positive 2-curvature (resp. positive h_4, positive h_4 and positive s_2) then $\phi_W(N) = 0$.

An important question in the same direction is Stolz's conjecture [16]:

Stolz Conjecture (1996). *If N is a string manifold and admits a positive Ricci curvature metric, then $\phi_W(N) = 0$.*

At this time no counterexamples are known to this conjecture and the conjecture is proven to be true for some classes. However, these classes admit also metrics with different positivity properties, for instance metrics with positive p-curvature, and so it may be possible that Stolz conjecture is true for positive p-curvature.

Let us note here that all the known constructions of string manifolds with positive p-curvature through group actions and Riemannian submersions [10], or through surgeries [8] have vanishing Witten genus. This is a consequence of a result due to Dessai, Höhn and Liu [5, 12] where they prove the vanishing of the Witten genus on connected string manifolds with non-trivial smooth S^3-actions and of another related result of Dessai [5]. The later asserts the vanishing of the Witten genus on any smooth fibre bundle of closed oriented manifolds provided the fibre is a string manifold and the structure group is a compact connected semi-simple Lie group which acts smoothly and non-trivially on the fibre.

6. Witten genus and its kernel in $\Omega_*^{\text{string}} \otimes \mathbf{Q}$

6.1. **Cayley projective plane.** Here we recall necessary facts concerning the Cayley projective plane $\mathbb{C}\mathbf{aP}^2$. We follow the constructions due to A. Dessai [5]. Let F_4 denote the 52-dimensional compact simple sporadic Lie group. It is well-known that F_4 contains a group isomorphic to $Spin(9)$ which is unique up to inner automorphism of the ambient group F_4. We choose such a subgroup and identify it with $Spin(9)$. Then we can identify the Cayley projective plane $\mathbb{C}\mathbf{aP}^2$ with the homogeneous space $F_4/Spin(9)$. This is 7-connected smooth manifold with the cohomology ring $H^*(\mathbb{C}\mathbf{aP}^2; \mathbf{Z}) \cong \mathbf{Z}[z]/z^3$, where $z \in H^8(\mathbb{C}\mathbf{aP}^2; \mathbf{Z})$ is a generator. In particular, $\mathbb{C}\mathbf{aP}^2$ is a fiber of the fiber bundle

$$BSpin(9) \to BF_4$$

induced by the embedding $Spin(9) \subset F_4$. The bundle $BSpin(9) \to BF_4$ is a universal *geometric* $\mathbb{C}\mathbf{aP}^2$*-bundle.*

Let L be a smooth manifold, $\dim L = \ell$, and $f : L \to BF_4$ be a map. Then one obtains the induced bundle with the fiber $\mathbb{C}\mathbf{aP}^2$ and structure group F_4:

$$\begin{tikzcd} W \arrow[r, "f^*"] \arrow[d, "\pi"] & BSpin(9) \arrow[d] \\ L \arrow[r, "f"] & BF_4 \end{tikzcd}$$

Let $T_{\mathrm{spin}} \subset Spin(9)$ be the maximal torus covering the maximal torus $T_{\mathrm{so}} \subset SO(9)$. It is convenient to choose a basis $\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \hat{\xi}_4$ of the Lie algebra of T_{so} which is also a basis for the Lie algebra of T_{spin}. Then the integral lattice in $\mathbf{R}^4$ which provides the universal cover of the torus T_{spin} is given by $a_1\hat{\xi}_1 + \cdots + a_4\hat{\xi}_4$, where the sum $a_1 + \cdots + a_4$ of integers is even. Let $\hat{\xi}$ be a generator of the Lie algebra of S^1, and $v : S^1 \to T_{\mathrm{spin}}$ be such a map for which the differential dv takes $\hat{\xi}$ to $2\hat{\xi}_1$. Then the composition

$$\hat{v} : S^1 \xrightarrow{v} T_{\mathrm{spin}} \longrightarrow Spin(9) \longrightarrow F_4$$

induces a map $B\hat{v} : BS^1 \to BF_4$. We obtain the following diagram of fiber bundles:

$$\begin{tikzcd} E \arrow[r, "(B\hat{v})^*"] \arrow[d, "\pi"] & BSpin(9) \arrow[d] \\ BS^1 \arrow[r, "B\hat{v}"] & BF_4 \end{tikzcd}$$

where the bundle $\pi : E \to BS^1$ has the fiber $\mathbb{C}\mathbf{aP}^2$ and the structure group is reduced from the group F_4 to its subgroup $v(S^1) \subset F_4$.

Then one can choose a subgroup S^3 of the centralizer of the group $v(S^1)$ in F_4 so that S^3 acts nontrivially on the orbit space $\mathbb{C}\mathbf{aP}^2 = F_4/Spin(9)$. A particular choice is given by the subgroup $S^3 \cong Spin(3) \subset Spin(9)$ which covers the subgroup

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & SO(3) \end{array}\right) \subset SO(9)$$

under the canonical covering map $Spin(9) \to SO(9)$. Then the subgroup S^3 commutes with the structure group $v(S^1)$ of the fiber bundle $\pi : E \to BS^1$. Thus we obtain a nontrivial action of S^3 along the fibers $\mathbb{C}\mathbf{aP}^2 = F_4/Spin(9)$ of the total space E.

Assume that a map $f : L \to BF_4$ is given by a composition $L \xrightarrow{h} BS^1 \xrightarrow{Bv} BF_4$. Then the geometric $\mathbb{C}\mathbf{aP}^2$-bundle $W \to L$ is given by the diagram of fiber bundles

$$\begin{tikzcd} W \arrow[r, "h^*"] \arrow[d] & E \arrow[r, "(B\hat{v})^*"] \arrow[d, "\pi"] & BSpin(9) \arrow[d] \\ L \arrow[r, "h"] & BS^1 \arrow[r, "B\hat{v}"] & BF_4 \end{tikzcd} \tag{9}$$

In particular, the structure group of the bundle $W \to L$ is reduced to $v(S^1) \subset F_4$, and there is a non-trivial fiber-wise action of S^3 on W. This construction leads to the following result:

Proposition 6.1. (A. Dessai, [5, Proposition 5.2]) *There exist oriented manifolds M^{4k} such that*

$$\Omega_*^{\mathrm{so}} \otimes \mathbf{Q} \cong \mathbf{Q}[x_4, \ldots, x_{4k}, \ldots],$$

where $x_{4k} = [M^{4k}]_{\mathrm{so}}$, $k = 1, 2, \ldots$, and M^{4k} satisfy the following conditions:

(a) *M^{4k} is a simply connected spin manifold for all $k \geq 1$;*
(b) *M^{4k} is a string manifold for all $k \geq 2$;*
(c) *M^{4k} is a total space of a geometric $\mathbb{C}\mathbf{aP}^2$-bundle with structure group S^1 and non-trivial S^3-action along the fibers for $k \geq 4$.*

We recall key points on the construction of manifolds M^{4k} given by A. Dessai [5]. The manifold M^4 could be chosen as the K_3-surface given by the quartic $x_0^4 + \cdots + x_3^4 = 0$ in $\mathbf{CP}^3$, and the manifolds M^8 and M^{12} as almost parallelizable manifolds with non-vanishing top Pontyagin class. These manifolds can be constructed by means plumbing, see [7]. The manifold $M^{16} = \mathbb{C}\mathbf{aP}^2$ is the Cayley projective plane. For $k \geq 5$, the manifold M^{4k} could be chosen as a total space of geometric $\mathbb{C}\mathbf{aP}^2$-bundle over a complete intersection $L_{k-4} \subset \mathbf{CP}^{2k-4}$ of complex dimension $2k-8$. The manifold L_{k-4} comes together with a nontrivial class $c \in H^2(L_{k-4}; \mathbf{Z})$ which is the first Chern class of the restriction of the dual Hopf bundle over $\mathbf{CP}^{2k-4}$. Then for carefully chosen integer a_{k-4}, the class $a_{k-4}c \in H^2(L_{k-4}; \mathbf{Z})$ gives a map $h_k : L_{k-4} \to BS^1$ such that the induced bundle $W \to L_{k-4}$ given by (9), where we let $L = L_{k-4}$ and $h = h_k$, is a geometric $\mathbb{C}\mathbf{aP}^2$-bundle with structure group S^1 and non-trivial S^3-action along the fibers.

Let M be a spin manifold, $\dim M = 4k$. Then the $\hat{A}(M)$ is well-defined and coincides with the index of the standard Dirac operator on M. For any real vector bundle V over M, we denote by $\hat{A}(M; V)$ the index of the Dirac operator on M twisted by the complexified vector bundle $V \otimes \mathbf{C}$. A *total symmetric power* $S_t(V)$ of a vector bundle V is given as a series

$$S_t(V) := 1 + S^1(V)t + S^2(V)t^2 + \cdots ,$$

where $S^j(V)$ is the j-th symmetric power of V and t is an indeterminate varaible. Consider the tensor product

$$\mathbb{S}(V) := \bigotimes_{m=1}^{\infty} S_{q^m}(V) = 1 + Vq + \big(S^2(V) \oplus V\big)\, q^2 + \big(S^3(V) \oplus (V \otimes V) \oplus V\big)\, q^3 + \cdots ,$$

see [19, Section 2]. Then the *Witten* genus $\phi_W(M)$ (where $\dim M = 4k$) is defined as the series

$$\begin{aligned} \phi_W(M) &= \hat{A}(M; \mathbb{S}(V)) \cdot \prod_{n=1}^{\infty} (1-q^n)^{4k} \\ &= \big(1 + Vq + \big(S^2(V) \oplus V\big)\, q^2 + \big(S^3(V) \oplus (V \otimes V) \oplus V\big)\, q^3 + \cdots\big) \cdot \prod_{n=1}^{\infty} (1-q^n)^{4k} \end{aligned}$$

see [19, Section 2] or [5, Section 2]. It is easy to see that $\phi_W(M) \in \mathbf{Z}[[q]]$, and $\hat{A}(M)$ is the constant term of the series $\phi_W(M)$. In particular, the Witten genus defines the homomorphism

$$\phi_W : \Omega_*^{\mathrm{string}} \longrightarrow \mathbf{Z}[[q]].$$

Proposition 6.1 implies the following result

Corollary 6.2. (A. Dessai, [5])

(1) *There is an isomorphism* $\Omega_*^{\mathrm{string}} \otimes \mathbf{Q} \cong \mathbf{Q}[x_8, x_{12}, x_{16}, \ldots, x_{4k}, \ldots]$, *where* $x_{4k} = [M^{4k}]_{\mathrm{string}}$, *and the string manifolds* M^{4k} *are as in Proposition* 6.1.
(2) *The kernal* $(\mathrm{Ker}\ \phi_W)\otimes\mathbf{Q} \subset \Omega_*^{\mathrm{string}}\otimes\mathbf{Q}$ *coincides with the ideal generated by the elements* x_{4k}, $k \geq 4$.
(3) *If* $x \in \mathrm{Ker}\ \phi_W \subset \Omega_*^{\mathrm{string}}$, *then some multiple of* x *can be realized as a total space of a geometric* $\mathbb{C}\mathbf{aP}^2$*-bundle.*

We emphasize the the string manifolds M^8 and M^{12} have non-trivial Witten genus just because $\hat{A}(M^8)$ and $\hat{A}(M^{12})$ are non-zero by construction.

7. Further Applications: Proof of Theorem A′ and Corollary B

The previous corollary 6.2 asserts in particular that if N is a string manifold with vanishing Witten genus then a non-zero multiple of N is string cobordant to a string manifold which is the total space of a $\mathbb{C}\mathbf{aP}^2$ bundle with structure group S^1 and non-trivial S^3-action along the fibres.

On the other hand the Cayley projective plane $\mathbb{C}\mathbf{aP}^2$ has dimension 16 and positive sectional curvature. In particular, using a result of [10], the total spaces of $\mathbb{C}\mathbf{aP}^2$ bundles have positive p-curvature for $0 \leq p \leq 14$ (and as well positive h_4-curvature). Corollary B results therefore immediately from Theorem B.

Next, we prove Theorem A′. Let N be 7-connected and non-Fivebrane compact manifold of dimension ≥ 15. Assume that N is string-cobordant to a manifold M which carries a metric with positive 6-curvature. Proposition 3.9 shows that the manifold N can then be obtained by performing surgeries on M of codimension $\geq 9 \geq 6+3$. Theorem 4.1 implies then that N carries a metric of positive 6-curvature.

Finally, the manifold N is 7-connected so it is a string manifold. Since the Witten genus of N is zero then by corollary 6.2, a non-zero multiple of N is string cobordant to a string manifold which is the total space of a $\mathbb{C}\mathbf{aP}^2$ bundle. As above the total spaces of $\mathbb{C}\mathbf{aP}^2$ bundles have positive 6-curvature, we deduce therefore from the first part of this theorem that some multiple $N\sharp\ldots\sharp N$ carries a metric of positive 6-curvature.

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